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Generalized Abelian Higgs models with self-dual vortices

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Abstract. We present a hierarchy of generalized Abelian Higgs models on $\mathbb{R}_2$, descended from the hierarchy of generalized Yang–Mills (GYM) systems on $\mathbb{R}_{4p}$, where the $p=1$ member is the usual Abelian Higgs model. We also study the vortex number n radial self-dual solutions numerically.

1. Introduction

The main purpose of the present work is to present a hierarchy of Abelian Higgs models generalizing the normal Abelian Higgs model [1, 2]. The qualitative features of the vortices in these models are the same as those of the vortices of the usual Abelian Higgs model. Of physical interest is the quantitative difference between vortices of different members of the hierarchy. Our work therefore supplies models which can accommodate vortices, like those in superconductors, with qualitatively the same but quantitatively different features.

The advantage of our construction is that the models are obtained by descent from higher-dimensional gauge field models which are endowed with simple geometric and topological properties. As a result the descendant models are guaranteed to have topologically stable solutions, whose vortex number is related to the magnetic flux just like that of the Abelian Higgs model. This is in contrast to a different generalization due to Lohe [3].

The Abelian Higgs model [1] on $\mathbb{R}_2$ has played an important role in physics in recent years, in particular as it allows for Nielson–Olesen [2] vortices in $(2+1)$ dimensions. The Lagrangian density on $\mathbb{R}_2$ is given by

$$\mathscr{L}^{(1)}=F_{ij}^2+2|D_i\varphi|^2+\frac{\lambda}{2}(\eta^2-\phi^2)^2 \tag{1.1}$$

where $F_{ij}=\partial_iA_j-\partial_jA_i$, and the Abelian gauge potential A_i interacts minimally with the complex scalar Higgs field φ via $D_i\varphi=\partial_i\varphi+\mathrm{i}A_i\varphi$. The Higgs field, whose modulus is ϕ, interacts with itself, via a symmetry-breaking potential. The field ϕ asymptotically tends to η for the topologically stable vortex solutions [1].

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The topological stability follows from the inequalities

$$[F_{ij}-\tfrac{1}{2}\varepsilon_{ij}(\eta^2-|\varphi|^2)]^2\geqslant 0 \tag{1.2a}$$

$$|D_i\varphi-\mathrm{i}\varepsilon_{ij}D_j\varphi|^2\geqslant 0 \tag{1.2b}$$

which can be rewritten as

$$F_{ij}^2+\tfrac{1}{2}(\eta^2-\phi^2)^2\geqslant\eta^2\varepsilon_{ij}\partial_iA_j-\varepsilon_{ij}\phi^2F_{ij} \tag{1.3a}$$

$$2|D_i\varphi|^2\geqslant-2\mathrm{i}\varepsilon_{ij}\partial_i(\varphi D_j\varphi^*)+\varepsilon_{ij}\phi^2F_{ij}. \tag{1.3b}$$

It follows from (1.3*a*) and (1.3*b*), then, that

$$\mathscr{L}^{(1)}\geqslant 2\varepsilon_{ij}\partial_i(\eta^2A_j-\mathrm{i}\varphi D_j\varphi^*) \tag{1.4}$$

provided that $\lambda>0$.

Thus, the volume integral of $\mathscr{L}^{(1)}$ is bounded from below by the 'surface' integral of the right-hand side of (1.4), where only the first term contributes, and is the topological charge. The second term on the right-hand side of (1.4) does not contribute to the 'surface' integral because $D_j\varphi$ decays like F_{ij} asymptotically, according to the *finite action* conditions, which is too fast, given that A_j decays like $1/r$. In the special case when $\lambda=1$, the inequality (1.3) can be saturated [1, 4], and the action integral becomes proportional to the vortex number

$$n=\frac{1}{2\pi}\int F_{12}\,\mathrm{d}^2x \tag{1.5}$$

which is just the non-vanishing contribution of the integral of the right-hand side of (1.4). In this case, the inequalities (1.2*a*) and (1.2*b*) become the Bogomol'nyi equalities [4], or the self-duality equations, the general solutions to which are discussed in [1]. The stress tensor for these self-dual multivortex solutions [1] vanishes identically, and the latter therefore represent non-interacting vortices in the plane.

In this paper, we present a hierarchy of Abelian Higgs models on $\mathbb{R}_2$, generalizing the model described by (1.1), in complete analogy to the hierarchy of the scale-invariant generalized Yang–Mills (GYM) models [5], *vis-à-vis* the usual YM model. To this end, we recall the scale-invariant GYM models on 4p dimensions

$$\mathscr{L}^{(p)}_{\mathrm{GYM}}=\mathrm{tr}\,F(2p)^2 \tag{1.6a}$$

$$F(2p)=F\wedge F\wedge\ldots\wedge F,\ p\text{ times} \tag{1.6b}$$

where the $p=1$ case is the usual YM system.

To explain this analogy, we start by recalling [1, 6] that the usual Abelian Higgs model on $\mathbb{R}_2$, with the coupling constant $\lambda=1$, is the residual subsystem of the SU(2) YM model on $\mathbb{R}_2\times S^2$, after the latter is subjected to dimensional reduction [7]. The topological density on the right-hand side of (1.4) is then the residual density resulting from the dimensional reduction [8] of the second Chern–Pontryagin (C-P) density $\mathrm{tr}\,F\wedge F$ on $\mathbb{R}_2\times S^2$. (The second C-P charge supplies the topological lower bound on the YM action.) Similarly, subjecting the scale-invariant, chiral $SO(4p)$ GYM action (1.6*a*) on $\mathbb{R}_2\times S^{4p-2}$ to dimensional reduction [7, 8], we find the *p*th member of the hierarchy of generalized Abelian Higgs models. For $p\geqslant 2$, and allowing for a coupling constant λ (>0) as in (1.1), the Lagrangian on $\mathbb{R}_2$ is

$$\mathscr{L}^{(p)} = (\eta^2-\phi^2)^{2(p-2)}\{4p(2p-1)(2p-2)!^2[(\eta^2-\phi^2)F_{ij} - \mathrm{i}(p-1)D_{[i}\varphi D_{j]}\varphi^*]^2 + 2p(\eta^2-\phi^2)^2|D_i\varphi|^2 + 4\lambda(2p-1)^2(\eta^2-\phi^2)^4\}. \tag{1.7}$$

We see that the coefficient of the highest power of η^2 in (1.7) is in fact nothing other than the usual Abelian Higgs model. The hierarchy (1.7) therefore consists of the usual Abelian Higgs model augmented by Skyrme-like terms. Note that for $p>1$ the Lagrangians (1.7) differ only in the power of the overall factor $(\eta^2-\phi^2)$ and in the coefficients of the different terms. In this sense there are only two qualitatively different members of this hierarchy, namely that for $p=1$ and those for $p>1$.

The discussion of formulae like (1.7) and the derivation of the corresponding topological charge densities is the subject of the next section. The topological charge providing the lower bound for the action of (1.7) is the residual charge arising from the dimensional reduction [8] of the $2p$th C-P charge on $\mathbb{R}_2\times S^{4p-2}$. This will be presented below in section 2. In section 3, the self-duality equations are studied numerically for the vortex number n radial field configurations corresponding to the $\lambda=1$ case. Section 4 contains our summary and conclusion.

2. The hierarchy of models

The derivation of (1.7) proceeds straightforwardly, employing the dimensional reduction formalism developed in [8]. The latter is based on the calculus of symmetric gauge fields formulated by Schwartz *et al* [9]. Since examples of such residual Lagrangians are treated in detail in [10] and [11] already, here we just quote our results. Subjecting the pth member of the GYM hierarchy (1.6*a*) on $\mathbb{R}_2\times S^{4p-2}$ to dimensional reduction, we find (1.7), with $\lambda=1$. In general we can consider $\lambda\neq 1$ ($\lambda>0$), in which cases the Bogomol'nyi inequalities to be given below cannot be saturated.

The main task to be addressed in this section is to show that the action of (1.7) is bounded from below by a topologically non-trivial charge. From the results of [8], we know that this must be the case. This is not a surprising result, since the action of the GYM system (1.6*a*) is bounded from below by the $2p$th C-P charge, and both (1.7) and the quantity bounding it from below are descended from the former densities, respectively, by a strict imposition of symmetries [8, 9].

Technically, our objective can be best achieved by exploiting the Bogomol'nyi bounds for each p. These in turn are derived by dimensional reduction of the (generalized) self-duality equations pertaining to (1.6), which saturate the inequality

$$\mathrm{tr}[F(2p) - {}^*F(2p)]^2 \geqslant 0. \tag{2.1}$$

It turns out that subjecting (2.1) to dimensional reduction over $\mathbb{R}_2\times S^{4p-2}$ always gives rise to *two* real inequalities, namely

$$(\eta^2-\phi^2)^{2(p-2)}\{[(\eta^2-\phi^2)F_{ij} - \mathrm{i}(p-1)D_{[i}\varphi D_{j]}\varphi^*] - (2p-1)\varepsilon_{ij}(\eta^2-\phi^2)^2\}^2 \geqslant 0 \tag{2.2a}$$

$$(\eta^2-\phi^2)^{2(p-1)}|D_i\varphi - \mathrm{i}\varepsilon_{ij}D_j\varphi|^2 \geqslant 0. \tag{2.2b}$$

The first inequality of (2.2) is the square of a real quantity, while the second is the modulus square of a complex quantity. Thus, saturating (2.2*a*) and (2.2*b*) would lead to the Bogomol'nyi equations

$$(\eta^2-\phi^2)^{(p-2)}\{[(\eta^2-\phi^2)F_{ij}-\mathrm{i}(p-1)D_{[i}\varphi D_{j]}\varphi^*]-(2p-1)\varepsilon_{ij}(\eta^2-\phi^2)^2\}=0 \qquad (2.3a)$$

$$(\eta^2-\phi^2)^{(p-1)}(D_i\varphi-\mathrm{i}\varepsilon_{ij}D_j\varphi)=0. \qquad (2.3b)$$

We shall be concerned with the solutions of (2.3*a*) and (2.3*b*) in the next section. Here, we proceed to show that (1.7) is bounded from below by virtue of the inequalities (2.2*a*) and (2.2*b*). For simplicity of presentation, we shall carry out the demonstration of this topological lower bound for the $p=2$ and $p=3$ members of the hierarchy only. We do not expect that this restriction sacrifices any generality since there are only two qualitatively different members of the hierarchy, namely $p=1$ and $p>1$. Both the action density (1.7) and the Bogomol'nyi equation (2.3) describe the usual Abelian Higgs model for $p=1$, the first member of the hierarchy. For $p=2$, the $U(1)$ kinetic term and the Higgs self-interaction potential are appreciably different from the $p=1$ case. For $p\geqslant 3$, these formulae differ from those of the $p=2$ case only through the multiplicative factor $(\eta^2-\phi^2)^{2(p-2)}$ and $(\eta^2-\phi^2)^{(p-2)}$ in (1.7) and (2.3*a*) respectively.

From the inequalities (2.2*a*) and (2.2*b*) we see that the linear combination of the *square* terms yields the action density (2.7), which is bounded from below by the corresponding linear combination of the *cross-terms*. The latter must be shown to be total divergences. We list these two cross-terms, $\rho_1^{(p)}$ and $\rho_2^{(p)}$, for $p=2$ and $p=3$, respectively:

$$\rho_1^{(2)}=6\varepsilon_{ij}[\eta^6F_{ij}-3\phi^2(\eta^4-\eta^2\phi^2+\tfrac{1}{3}\phi^4)F_{ij}-2\mathrm{i}(\eta^2-\phi^2)^2D_i\varphi D_j\varphi^*] \qquad (2.4a)$$

$$\rho_2^{(2)}=\mathrm{i}\varepsilon_{ij}(\eta^2-\phi^2)^2D_i\varphi D_j\varphi^* \qquad (2.4b)$$

for $p=2$, and

$$\rho_1^{(3)}=10\varepsilon_{ij}[\eta^{10}F_{ij}-5\phi^2(\eta^8-2\eta^6\phi^2+2\eta^4\phi^4-\eta^2\phi^6+\tfrac{1}{5}\phi^8)F_{ij}-4(\eta^2-\phi^2)^4\mathrm{i}D_i\varphi D_j\varphi^*] \qquad (2.5a)$$

$$\rho_2^{(3)}=\mathrm{i}\varepsilon_{ij}(\eta^2-\phi^2)^4D_i\varphi D_j\varphi^* \qquad (2.5b)$$

for $p=3$. Note that the second type of cross-terms $\rho_2^{(p)}$ occur in the first, $\rho_1^{(p)}$.

We can now verify the following identities:

$$\rho_2^{(2)}=\mathrm{i}\varepsilon_{ij}\partial_i[(\eta^4-\eta^2\phi^2+\tfrac{1}{3}\phi^4)\varphi D_j\varphi^*]-\tfrac{1}{2}\varepsilon_{ij}\phi^2(\eta^4-\eta^2\phi^2+\tfrac{1}{3}\phi^4)F_{ij} \qquad (2.6a)$$

$$\rho_2^{(3)}=\mathrm{i}\varepsilon_{ij}\partial_i[(\eta^8-2\eta^6\phi^2+2\eta^4\phi^4-\eta^2\phi^6+\tfrac{1}{5}\phi^8)\varphi D_j\varphi^*]-\tfrac{1}{2}\varepsilon_{ij}\phi^2(\eta^8-2\eta^6\phi^2+2\eta^4\phi^4-\eta^2\phi^6+\tfrac{1}{5}\phi^8)F_{ij}. \qquad (2.6b)$$

Finally, choosing the linear combinations $(\rho_1^{(2)}+24\rho_2^{(2)})$ and $(\rho_1^{(3)}+50\rho_2^{(3)})$, respectively, we find that the action density $\mathscr{L}^{(p)}$ for $p=2$ and 3 is bounded from below by the following *topological* densities:

$$\partial_i\Omega_i^{(2)}=12\partial_i\varepsilon_{ij}[\eta^6A_j-3\mathrm{i}(\eta^4-\eta^2\phi^2+\tfrac{1}{3}\phi^4)\varphi D_j\varphi^*] \qquad (2.7a)$$

$$\partial_i\Omega_i^{(3)}=20\partial_i\varepsilon_{ij}[\eta^{10}A_j-\tfrac{9}{2}\mathrm{i}(\eta^8-2\eta^6\phi^2+2\eta^4\phi^4-\eta^2\phi^6+\tfrac{1}{5}\phi^8)\varphi D_j\varphi^*]. \qquad (2.7b)$$

Note the similarity of (2.7*a*) and (2.7*b*) with the topological density on the right-hand side of (1.4), for $p=1$. In all these cases, the only terms that contribute to the surface integrals are the first terms, for reasons explained in section 1. Thus we see that for $p=1, 2, 3$ the topological charge for the hierarchy of these models is always the *vortex number* (1.5). It is clear that, starting from (2.2*a*) and (2.2*b*), results similar to (2.7*a*) and (2.7*b*) can be derived straightforwardly for arbitrary p.

3. Radial self-dual solutions

We restrict our considerations of the self-duality equations here to the radial excitation field configurations

$$A_i = \varepsilon_{ij}\hat{x}_j \frac{a(r)-n}{r} \tag{3.1a}$$

$$\varphi = \eta g(r)\, \mathrm{e}^{-in\theta} \tag{3.1b}$$

with vortex number n. We shall study the solutions of the self-dual equations (2.3*a*) and (2.3*b*). Using the ansatz (3.1), this hierarchy of Bogomol'nyi equations (2.3*a*) and (2.3*b*) reduces to the following hierarchy of ordinary nonlinear differential equations:

$$(1-g^2)\frac{\mathrm{d}a}{\mathrm{d}r} = \frac{2}{r}(p-1)a^2g^2 - r(2p-1)(1-g^2)^2 \tag{3.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}g = \frac{1}{r}ag. \tag{3.2b}$$

Since we are unable to integrate (3.2*a*) and (3.2*b*) explicitly, we cannot find the full solutions in closed form. Instead we find the behaviour of the solutions for $r \ll 1$ and $r \gg 1$, with parameters to be determined numerically by the shooting method. The full solutions are studied numerically and, for $p=2$ and $n=1, 2$, the functions are plotted.

For large r, we want solutions with asymptotic behaviour $a(r)\to 0$ and $g(r)\to 1$ as $r\to\infty$. If $\delta \equiv [1-g(r)] \ll 1$, (3.2) are approximated by

$$a = -r\frac{\mathrm{d}\delta}{\mathrm{d}r} \tag{3.3a}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\delta^p}{\mathrm{d}r}\right) = 2p(2p-1)\delta^p. \tag{3.3b}$$

These equations have solutions with the required exponential decay, namely

$$g = 1 - C[K_0(\sqrt{2p(2p-1)}\,r)]^{1/p} \tag{3.4a}$$

$$a = C\sqrt{\frac{2(2p-1)}{p}}\, r[K_0(\sqrt{2p(2p-1)}\,r)]^{(1-p)/p}K_1(\sqrt{2p(2p-1)}\,r). \tag{3.4b}$$

Near the origin ($r\to 0$), we want solutions with asymptotic behaviour $a(r)\to n$ and $g(r)\to 0$. If, for small r, we attempt a power series solution, we obtain

$$a = 1 + [(p-1)C_1^2 + \tfrac{1}{2}(1-2p)]r^2 + \mathrm{o}(r^2) \tag{3.5a}$$

$$g = C_1 r + [\tfrac{1}{2}(p-1)C_1^3 + \tfrac{1}{4}(1-2p)C_1]r^3 + \mathrm{o}(r^3) \tag{3.5b}$$

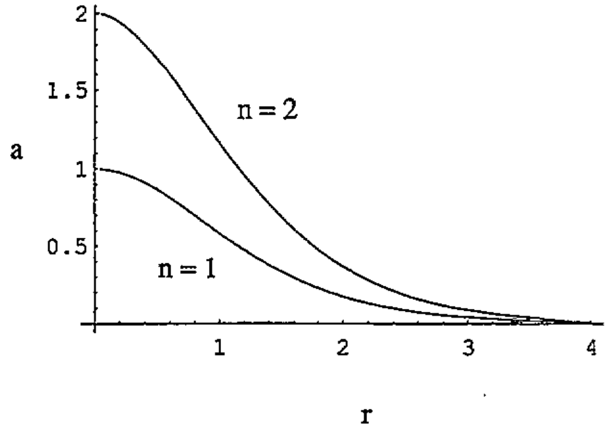

Figure 1. a as a function of r for $p=2$ and $n=1, 2$.

for $n=1$. For $n \geqslant 2$, we find

$$a = n + \tfrac{1}{2}(1-2p)r^2 + (p-1)nC_n^2 r^{2n} + o(r^{2n}) \tag{3.6a}$$

$$g = C_n r^n + \tfrac{1}{4}(1-2p)C_n r^{n+2} + o(r^{n+2}). \tag{3.6b}$$

Higher-order terms can easily be calculated recursively.

We have studied (3.2) numerically and have found solutions with the correct asymptotic behaviour. This numerical result can be understood as follows. If we choose $C_n (n = 1, 2, 3, \ldots)$ too small, $g(r)$ will increase so slowly that $a(r)$ becomes negative and $g(r)$ starts decreasing instead of approaching the value 1. If we choose C_n too large, $g(r)$ will approach the value 1 fast, and the first term on the right-hand side of (3.2a) will dominate the second term and prevent $a(r)$ from decreasing and approaching the value 0. The value of C_n separating the two regimes gives the solution we seek. We have plotted the functions $a(r)$ and $g(r)$ in figures 1 and 2, respectively, for $p=2$ and $n = 1, 2$. The corresponding values for C_n we found are $C_1 = 0.95$ and $C_2 = 0.92$.

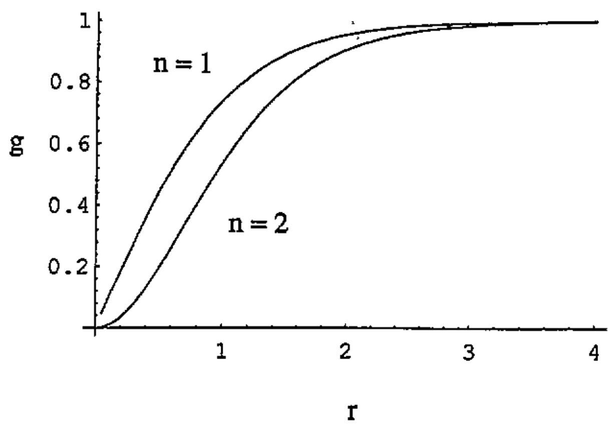

Figure 2. g as a function of r for $p=2$ and $n=1, 2$.

4. Summary and conclusions

We have presented a hierarchy of Abelian Higgs models on R_2 labelled by an integer p. The pth model of the hierarchy is that obtained from the GYM model on the $4p$-dimensional manifold $\mathbb{R}_2 \times S^{4p-2}$ by dimensional reduction. The $p=1$ member of this hierarchy is the usual Abelian Higgs model.

Apart from defining the actions (1.7) for these models we have supplied the topological inequalities (2.2) which guarantee the stability of the vortex solutions, and we have in particular considered the self-dual solutions satisfying the Bogomol'nyi equations (2.3). Restricting to the radial solutions of winding number n, we have found approximate solutions (3.4) for $r \gg 1$ and (3.5) for $r \ll 1$. The full solution has been found numerically for the $p=2$ model for the winding numbers $n=1$ and $n=2$, and this serves as our demonstration for the existence of the solutions.

We conclude that our hierarchy of generalized Abelian Higgs models is endowed with (self-dual) vortex solutions, whose qualitative properties are largely p independent. From the physical point of view it may be of some interest whether vortices of different models in this hierarchy may have quantitatively different properties. To this end, we have drawn the curves for the function $a(r)$ for the two models of the hierarchy labelled

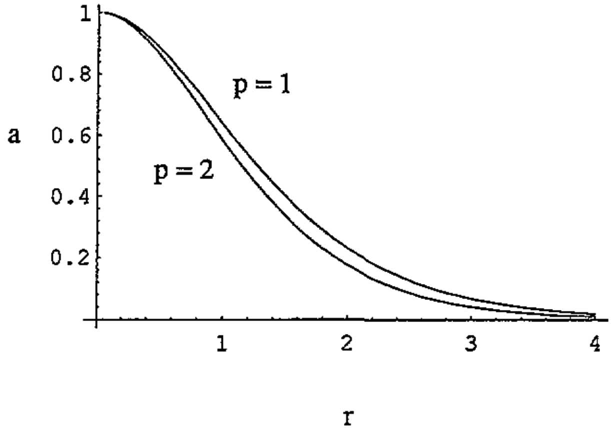

Figure 3. a as a function of r for $p=1$ and 2, for $n=1$.

by $p=1$ and 2, each with winding number $n=1$. This is given in figure 3, which exhibits the quantitative difference sought, namely that the $p=1$ and 2 models result in vortices of different widths, notwithstanding the fact that the same scale-breaking dimensional parameter η features in both models. It is conceivable that this distinction may prove to be relevant to the physics of the vortices.

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